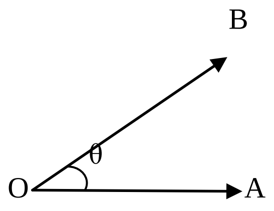
TRIGONOMETRY

In this chapter we intend to study an important branch of mathematics called 1trigonometryIt is the science of measuring angle of triangles, side of triangles.

Angle:-



Consider a ray OA if this ray rotate about its end point O and takes the position OB then we say that the angle ∠AOB has been generated.

Measure of angle: The measure of an angle is the amount of rotation from initial side to the terminal side.

NOTE:

Relation between degree and radian measurement π radians = 180 degree

radian measure $= \frac{17}{180} \times$ degree measure

degree measure $\frac{180}{\pi} \times$ radian measure

$$1° = 60'(60\ minutes)$$
$$1' = 60"(60\ seconds)$$

Example 1:

Find radian measure of 270°.

Solution:

Radian measure $= \frac{\pi}{180} \times 270 = \frac{3\pi}{2}$

Example 2:

Find degree measure of$\frac{5\pi}{9}$

Solution:

degree measure $= \frac{180}{\pi} \times \frac{5\pi}{9} = 100°$

Example 3:

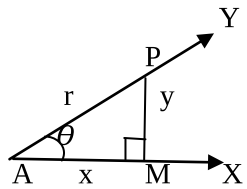
If the arcs of same length in two circles subtend angles of 60° and 75* at their centres. Find the ratio of their radii.

Solution:

$$60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c \text{ and}$$
$$75° = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$$
$$\therefore \frac{\pi}{3} = \frac{s}{r_1}\ and\ \frac{5\pi}{12} = \frac{s}{r_2}\left[\because\ \theta = \left(\frac{s}{r}\right)^c\right]$$
$$\Rightarrow \frac{\pi}{3}r_1 = s\ and\ \frac{5\pi}{12}r_2 = s$$
$$\Rightarrow \frac{\pi}{3}r_1 = \frac{5\pi}{12}r_2$$
$$\Rightarrow 4r_1 = 5r_2 \Rightarrow r_1 : r_2 = 5:4$$

Trigonometric ratios:

The most important task of trigonometry is to find the remaining side and angle of a triangle when some of its side and angles are given. This problem is solved by using some ratio of sides of a triangle with respect to its acute angle. These ratio of acute angle are called trigonometric ratio of angle. Let us now define various trigonometric ratio.



Consider an acute angle ∠YAX = θ with initial side AX and terminal side AY. Draw PM perpendicular from P on AX to get right angle triangle AMP. In right angle triangle AMP.

Base = AM = x

Perpendicular = PM = y and

Hypotenuse = AP = r.

$$r = \sqrt{x^2 + y^2}$$

We define the following six trigonometric Ratios:

$$\sin\theta = \frac{Perpendicular}{Hypotenuse} = \frac{y}{r}$$
$$\cos\theta = \frac{Base}{Hypotenuse} = \frac{x}{r}$$

$$\tan\theta = \frac{Perpendicular}{Base} = \frac{y}{x}$$
$$\operatorname{cosec}\theta = \frac{Hypotenuse}{Perpendicular} = \frac{r}{y}$$
$$\sec\theta = \frac{Hypotenuse}{Base} = \frac{r}{x}$$
$$\cot\theta = \frac{Base}{Perpendicular} = \frac{x}{y}$$

Important formula:-

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\operatorname{cosec}^2\theta - \cot^2\theta = 1$
3. $\sec^2\theta = \tan^2\theta = 1$

θ T-ratio	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\operatorname{cosec}\theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\cot\theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

5. $\sin(90° - \theta) = \cos\theta$
6. $\cos(90° - \theta) = \sin\theta$
7. $\tan(90° - \theta) = \cot\theta \Rightarrow \cot(90° - \theta) = \tan\theta$
8. $\operatorname{cosec}(90° - \theta) = \sec\theta$
9. $\sec(90° - \theta) = \operatorname{cosec}\theta$

RELATIONAMONGT – RATIONS

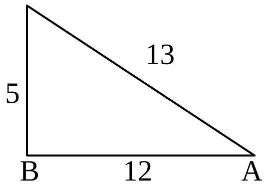
	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+tan^2\theta}}$	$\frac{1}{\sqrt{1+cot^2\theta}}$	$\frac{\sqrt{sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{1-sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{1}{\frac{\sqrt{cosec^2\theta-1}}{cosec\ \theta}}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-sin^2\theta}}$	$\frac{\sqrt{1-cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{sec^2\theta-1}$	$\frac{1}{\sqrt{cosec^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{sec^2\theta-1}}$	$\sqrt{cosec^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1-tan^2\theta}$	$\frac{\sqrt{1+cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\operatorname{cosec}\theta}{\sqrt{cosec^2\theta-1}}$

$\operatorname{cosec}\theta$	$\dfrac{1}{\sin\theta}$	$\dfrac{1}{\sqrt{1-cos^2\theta}}$	$\dfrac{\sqrt{1+tan^2\theta}}{\tan\theta}$	$\sqrt{1+cot^2\theta}$	$\dfrac{\sec\theta}{\sqrt{sec^2\theta-1}}$	$\operatorname{cosec}\theta$

Example 4:

In a ΔABC right angled at B if AB = 12, and BC = 5 find sin A and tan A, cos C and cot C

Solution: C



$$AC = \sqrt{(AB)^2 + (BC)^2}$$
$$= \sqrt{12^2 + 5^2}$$
$$= \sqrt{144 + 25}$$
$$= \sqrt{169} = 13$$

When we consider t-ratios of∠A we have

Base AB = 12

Perpendicular = BC = 5

Hypotenuse = AC = 13

$$\sin A = \frac{\text{Perpendicular}}{Hypotenuse} = \frac{5}{13}$$
$$\tan A = \frac{\text{Perpendicular}}{Base} = \frac{5}{12}$$

When we consider t-ratios of ∠C, we have

Base = BC = 5

Perpendicular = AB = 12

Hypotenuse = AC = 13

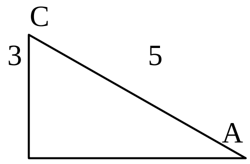
$$\cos C = \frac{Base}{Hypotenuse} = \frac{5}{13}$$
$$\cot C = \frac{Base}{Perpendicular} = \frac{5}{12}$$

Example 5:

In a right triangle ABC right angle at B the six trigonometric ratios of ∠C

Solution:

$$\sin A = \frac{Perpendicular}{Hypotenuse} = \frac{3}{5}$$



$$\text{Base} = \sqrt{(\text{Hypotenuse})^2 - (\text{Perpendicualr})^2}$$
$$= \sqrt{5^2 - 3^2}$$
$$= \sqrt{25-9} = \sqrt{16} = 4$$

Now

$$\sin C = \frac{BC}{AC} = \frac{4}{5}, \operatorname{cosec} C = \frac{5}{4}$$
$$\cos C = \frac{3}{5} = \frac{AB}{AC}, \sec C = \frac{5}{3}$$
$$\tan C = \frac{AB}{AC} = \frac{4}{3}, \cot C = \frac{3}{4}$$

Example 6:

Find the value of 2 $\sin^2 30°$ tan 60° - 3 $\cos^2 60°$ $\sec^2 30°$

Solution:

$$2\left(\frac{1}{2}\right)^2 \times \sqrt{3} - 3\left(\frac{1}{2}\right)^2 \times \left(\frac{2}{\sqrt{3}}\right)^2$$
$$= 2 \times \frac{1}{4} \times \sqrt{3} - 3 \times \frac{1}{4} \times \frac{4}{3} = \frac{\sqrt{3}}{2} - 1$$
$$= \frac{\sqrt{3}-2}{2}$$

Example 7:

Find the value θ sin $2\theta=\sqrt{3}$

Solution:

$$\sin 2\theta = \frac{\sqrt{3}}{2}$$

2θ = 60

θ = 30°

Example 8:

Find the value of x.

Tan 3x = sin 45° cos 45° + sin 30°

Solution:

$$\tan 3x = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$\Rightarrow \tan 3x = 1 \;\Rightarrow \tan 3x = \tan 45°$$

3x = 45°

X = 15°

Example 9:

If θ is an acute angle tan θ + cot θ=2 find the value of $\tan^7 + \theta + \cot^7$.

Solution:

$$\tan\theta + \cos\theta = 2$$

$$\tan\theta + \frac{1}{\tan\theta} = 2$$

$$\Rightarrow \tan^2\theta + 1 = 2\tan\theta$$

$$\Rightarrow \tan^2\theta - 2\tan\theta + 1 = 0$$

$$(\tan\theta - 1)^2 = 0$$

$$\tan\theta = 1$$

$$\theta = 45°$$

Now, $\tan^7\theta + \cot^7\theta$.

$$= \tan^7 45° + \cot^7 45°$$

$$= 1 + 1 = 2$$

Example 10:

Find the value of $\frac{\cos 37°}{\sin 53°}$

Solution:

We have

$$\frac{\cos 37°}{\sin 53°} = \frac{\cos(90° - 53°)}{\sin 53°} = \frac{\sin 53°}{\sin 53°} = 1$$

Example 11:

Find the value of

$$\frac{\sin 36°}{\cos 54°} - \frac{\sin 54°}{\cos 36°}$$

Solution:

We have

$$= \frac{\sin 36°}{\cos 54°} - \frac{\sin 54°}{\cos 36°}$$

$$= \frac{\sin(90° - 54°)}{\cos 54°} - \frac{\sin(90° - 36°)}{\cos 36°}$$

$$= \frac{\cos 54°}{\cos 54°} - \frac{\cos 36°}{\cos 36°}$$

$$= 1 - 1 = 0$$

Example 12:

Evaluate the cot 12° cot 38°cot 52° cot 60° cot 78°

Solution:

We have

cot 12° cot 38° cot 52° cot 60° cot 78°

$$= (\cot 12° \cot 78°)(\cot 38° \cot 52°)\cot 60°$$

$$= [\cot 12° \cot(90° - 12°)][\cot 38° \cot(90° - 38°)]\cot 60°$$

$$= [\cot 12° \tan 12°][\cot 38° \tan 38°]\cot 60°$$

$$= 1 \times 1 \times \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Example 13:

If tan 2θ = cot (θ + 6°), where 2θ and θ+ 6° are acute angles find the value of θ.

Solution:

We have

$$\tan 2\theta = \cot(\theta + 6°)$$

$$\cot(90° - 2\theta) = \cot(\theta + 6°)$$

$$90 - 2\theta = \theta + 6°$$

$$3\theta = 84°$$

$$\theta = 28°$$

Example 14:

Find the value of (l– $\sin^2\theta$) $\sec^2\theta$.

Solution:

We have,

$$(1 - \sin^2\theta)(\sec^2\theta)$$

$$= \cos^2\theta \sec^2\theta$$

$$= \cos^2\theta \times \frac{1}{\cos^2\theta}$$

= 1

Example 15:

$\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta}$find its valu

Solution:

We have

$$\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta}$$

$$= \frac{1 - \sin\theta + 1 + \sin\theta}{(1+\sin\theta)(1-\sin\theta)}$$

$$= \frac{2}{1 - \sin^2\theta} = \frac{2}{\cos^2\theta} = 2\sec^2\theta$$

Example 16:

Find the value of $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$

Solution:

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sqrt{\frac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$$

$$= \frac{1-\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta.$$

Example 17:

Find the value of [(1 + cot θ) - cosec θ] [1 + tan θ + sec θ]

Solution:

= (1+ cot θ- cosec θ) (1+ tan θ + sec θ)

$$= \left(1+\frac{\cos\theta}{\sin\theta}-\frac{1}{\sin\theta}\right)\left(1+\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}\right)$$

$$= \left(\frac{\sin\theta+\cos\theta-1}{\sin\theta}\right)\left(\frac{\cos\theta+\sin\theta+1}{\cos\theta}\right)$$

$$= \frac{(\sin\theta+\cos\theta)^2-1}{\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta}$$

$$= \frac{1+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2$$

Example 18:

If $\sin\theta = \frac{3}{5}$, find the value of $\sin\theta\cos\theta$.

Solution:

$$\sin\theta = \frac{3}{5}$$

$$\cos\theta = \sqrt{1-\sin^2\theta}$$

$$= \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

$$\sin\theta\times\cos\theta = \frac{3}{5}\times\frac{4}{5} = \frac{12}{25}$$

Example 19:

If $\cos\theta = \frac{1}{2}$, find the value if, $\frac{2\sec\theta}{1+\tan\theta}$

Solution:

$$\cos\theta = \frac{1}{2}$$

$$\sec\theta = 2$$

$$\frac{2\sec\theta}{1+\tan^2\theta} = \frac{2\sec\theta}{\sec^2\theta} = \frac{2}{\sec\theta} = \frac{2}{2} = 1$$

Example 20:

If $\tan\theta = \frac{12}{15}$, find the value of $\frac{1+\sin\theta}{1-\sin\theta}$

Solution:

$$\tan\theta = \frac{12}{5}$$

$$\sec\theta = \sqrt{1+\tan^2\theta}$$

$$= \sqrt{1+\left(\frac{12}{5}\right)^2} = \frac{13}{5}$$

$$\cos\theta = \frac{5}{13}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \frac{12}{13}$$

$$\text{thus } \frac{1+\sin\theta}{1-\sin\theta} = \frac{1+\frac{12}{13}}{1-\frac{12}{13}} = \frac{\frac{25}{13}}{\frac{1}{13}} = 25$$

Example 21:

If $\sin\theta = \frac{a}{\sqrt{a^2+b^2}} 0 < \theta < 90°$ find the value of $\tan\theta$

Solution:

$$\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$$

$$\cos\theta = \sqrt{1-\sin^2\theta}$$

$$\cos\theta = \sqrt{1-\frac{a^2}{a^2+b^2}} = \sqrt{\frac{b^2}{a^2+b^2}}$$

$$= \frac{b}{\sqrt{a^2+b^2}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{a}{\sqrt{a^2+b^2}}}{\frac{b}{\sqrt{a^2+b^2}}} = \frac{a}{b}$$

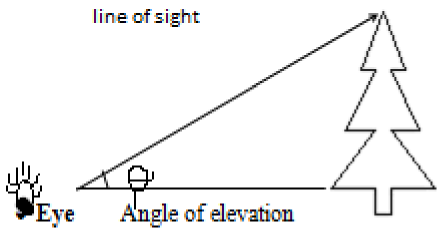
HEIGHT AND DISTANCE

Sometimes, we have to find the height of a tower, building, tree, distance of a ship, width of a river, etc. Though we cannot measure them easily, we can determine these by using trigonometric ratios.

Line of Sight

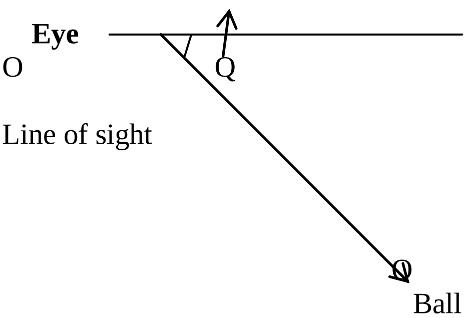
The line of sight or the line of vision is a straight line to the object we are viewing.

If the object is above the horizontal from the eye, we have to lift up our head to view the object. In this process, our eye move, through an angle. This angle is called the angle of elevation of the object.



If the object is below the horizontal from the eye, then we have no turn our head downwards no view the object. In this process, our eye move through an angle. This angle is called the angle of depression of the object.

Angle of depression



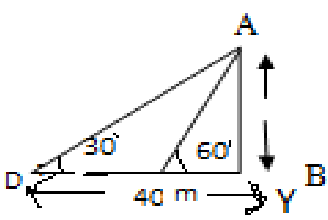
Example 22:

A person observed the angle of elevation of the top of a tower b 30°. He walked 40 m towards the foot of the tower along level ground and found the angle of elevation of the top of the tower as 60°. Find the height of tower.

Solution:

Let height of tower AB = x m and BC = y m, DC = 40 m.

n ΔABC,



$\frac{AB}{Bc}$ = tan 60° → $\frac{X}{Y}$ = $\sqrt{3}$y→ $\frac{X}{\sqrt{3}}$

Now In rt↓ ABD, $\frac{AB}{BD}$= tan 30°

⇨ $\frac{X}{40+y}$ =$\sqrt{3}$x = 40 + $\frac{X}{\sqrt{3}}$ [using (i)]

⇨ $3x$ = 40 $\sqrt{3}$+ x = 3x-x=40$\sqrt{3}$ = 2x= 40$\sqrt{3}$

x= 20$\sqrt{3}m$

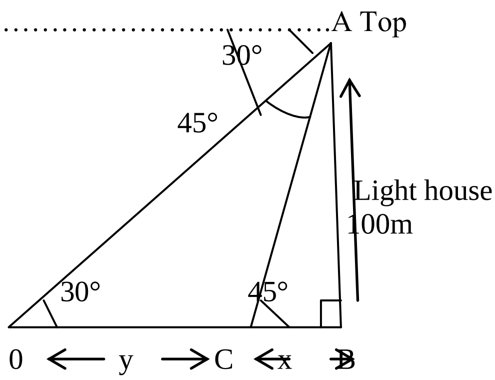
Example 23:

As observed from top of a light house 100 m. high above sea level, the angle of depression of a ship sailing directly toward it changes from 30° to 45°. The distance travelled by the ship during the period of observation is

Solution :

Let 'Y' be the required distance between two positions 0 and Cof the ship

In rt. ABC



cot 45° = $\frac{X}{100}$ = x = 100 ……….(i)

In A AOB, $\frac{y+x}{100}$ = cot 30°

⇨ y + x = 100 $\sqrt{3}$= y = 100$\sqrt{3} - x$

⇨ y = 100$\sqrt{3}$ – 100 [using (i)]

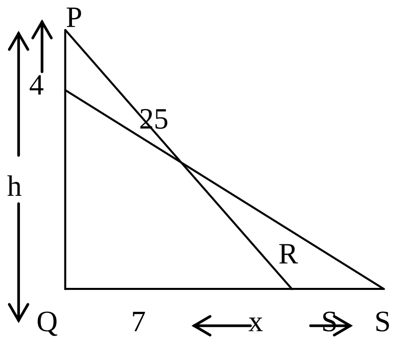
⇨ y = 100 ($\sqrt{3}$ – 1)

⇨ y = 100 (1.732 – 1) = 100 x 0.732 = 73.20m.

Example 24:

A 25 m long ladder is placed against a vertical wall of a building. The foot of the ladder is 7m from base of the building. If the top of the ladder slips 4m, then the foot of the Ladder will slide by how much distance.

Sol: Let the height of the wall be h.



Now, h = $\sqrt{25^2 - 7^2}$

= $\sqrt{576}$ = 24m

QS = $\sqrt{625 - 400}$

= $\sqrt{225} = 15m$

Required distance, X = (15-7) = 8m